



UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 1st Semester Examination, 2019

GE-MATHEMATICS

Time Allotted: 2 Hours

Full Marks: 60

*The figures in the margin indicate full marks.**Candidates should answer in their own words and adhere to the word limit as practicable.**All symbols are of usual significance.*

The question paper contains GE1, GE2, GE3, GE4 and GE5.
Candidates are required to answer any *one* from the *five* courses and
they should mention it clearly on the Answer Book.

GE1

CALCULUS, GEOMETRY, DIFFERENTIAL EQUATION

GROUP-A

1. Answer any *four* questions from the following: 3×4 = 12

(a) Evaluate $\int_0^1 x e^{-\sqrt{x}} dx$ using reduction formula. 3

(b) Find $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^{1/x}$. 3

(c) Determine the nature of the quadric 3

$$x^2 + y^2 + z^2 - yz - zx - xy - 3x - 6y - 9z + 21 = 0$$

(d) If $y = \frac{1}{x^2 + a^2}$, find y_n , where suffix n represents n^{th} derivative. 3

(e) $f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ 3

Discuss the continuity and differentiability of f at $x = 0$.

(f) Find the equation of the line $y = \sqrt{3}x$, when the axes rotated through an angle $\frac{\pi}{3}$. 3

GROUP-B

2. Answer any *four* questions from the following: 6×4 = 24

(a) If $y = a \cos(\log x) + b \sin(\log x)$, show that $x^2 y_{n+2} + (2n+1)x y_n + (n^2 + 1)y_n + 1 = 0$. 6

(b) Find the equations of the circle on the sphere $x^2 + y^2 + z^2 = 49$ whose centre is at the point $(2, -1, 3)$. 6

- (c) Find the envelope of the straight lines $\frac{x}{a} + \frac{y}{b} = 1$ when $a^n + b^n = c^n$. 6
- (d) Find the volume of the solid generated by revolving the cardioide $r = a(1 + \cos \theta)$ about initial line. 6
- (e) Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is $\frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$. 6
- (f) Show that $(a^2 + b^2)(x^2 + y^2) = (ax + by - ab)^2$ represents a parabola of latus rectum $\frac{2ab}{\sqrt{a^2 + b^2}}$. 6

GROUP-C

3. Answer any **two** questions from the following: 12×2= 24
- (a) (i) Find the points of inflexion on the curve $(\theta^2 - 1)r = a\theta^2$. 3
- (ii) Find the envelope of the ellipse $x = a \sin(\theta - \alpha)$, $y = b \cos \theta$ where α is the parameter. 3
- (iii) Prove that the equation $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ represents a pair of planes. Find the angle between them. 4+2
- (b) (i) Find the area bounded by one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ and the X-axis. 3
- (ii) If $y = x \log \left(\frac{x-1}{x+1} \right)$, prove that $y_n = (-1)^n (n-2)! \left[\frac{x-n}{(x+1)^n} - \frac{x+n}{(x+1)^n} \right]$. 3
- (iii) If a sphere touches the planes $2x + 3y - 6z + 14 = 0$ and $2x + 3y - 6z + 42 = 0$ and if its centre lies on the straight line $2x + z = 0$, $y = 0$, find the equation of the sphere. 6
- (c) (i) Find the equation of the cylinder which passes through the point $(3, -1, 1)$ and has the axis $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{1}$. 6
- (ii) Find the equation of the cone whose vertex is the origin and base is the circle $x = a$, $y^2 + z^2 = b^2$. 6
- (d) (i) Solve and find the singular solution of the equation $p^3 - 4xyp + 8y^2 = 0$. 6
- (ii) Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$. 6

GE2
ALGEBRA
GROUP-A

1. Answer any **four** questions from the following: 3×4 = 12
- (a) Prove that the determinant of an orthogonal matrix is ± 1 . 3
- (b) Show that the mapping $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^3$, $x \in \mathbb{R}$ is a bijection. 3
- (c) If $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, find the rank of the matrix $A + A^2$. 3
- (d) If $T: \mathbb{R}_{2 \times 2} \rightarrow \mathbb{R}_{2 \times 2}$ defined by $T(A) = \frac{1}{2}(A + A^t)$, $A \in \mathbb{R}_{2 \times 2}$, then show that T is linear and find $\text{Ker } T$, where $\mathbb{R}_{2 \times 2}$ is the set of all 2×2 real matrices. 3
- (e) If P is an $n \times n$ real orthogonal matrix with $\det P = -1$. Prove that $P + I_n$ is a singular matrix. 3
- (f) Prove that, $\log(3 + 4i) = \log 5 + (2n\pi + \tan^{-1} \frac{4}{3})i$. 3

GROUP-B

2. Answer any **four** questions from the following: 6×4 = 24
- (a) If $x = \log \tan\left(\frac{\pi}{4} + \frac{1}{2}y\right)$, then prove that $y = -i \log \tan\left(\frac{\pi}{4} + \frac{1}{2}ix\right)$. 6
- (b) If α, β, γ are the roots of the equation $2x^3 + 3x^2 - x - 1 = 0$, find the equation whose roots are $\frac{\alpha}{\beta + \gamma}, \frac{\beta}{\gamma + \alpha}, \frac{\gamma}{\alpha + \beta}$. 6
- (c) Prove that $1! \cdot 3! \cdot 5! \cdots (2n-1)! > (n!)^n$. 6
- (d) Prove that the linear mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that maps the basis vectors $(1, 2, 2), (2, 1, 2), (2, 2, 1)$ of $\mathbb{R}^3$ to the vectors $(0, 1, 1), (1, 0, 1), (1, 1, 0)$ respectively is one-one and onto. 6
- (e) Show that the mapping $f: S \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x}{1-|x|}$, $x \in S$ is a bijection. 6
- Determine f^{-1} where $S = \{x \in \mathbb{R} : -1 < x < 1\}$.
- (f) For what values of λ the following system of equations are consistent? 6
- $$\begin{aligned} x - y + z &= 1 \\ x + 2y + 4z &= \lambda \\ x + 4y + 6z &= \lambda^2 \end{aligned}$$

GROUP-C

3. Answer any *two* questions from the following: 12×2= 24

- (a) (i) If $\tan \log(x+iy) = a+ib$, where $a^2 + b^2 \neq 1$, then prove that 6

$$\tan \log(x^2 + y^2) = \frac{2a}{1 - a^2 - b^2}.$$

- (ii) If $\log \sin(\theta + i\phi) = \alpha + i\beta$, then prove that $2e^{2\alpha} = \cosh 2\phi - \cos 2\theta$. 6

- (b) (i) Find the eigen values and the corresponding eigen vectors of the following 6

real matrix:
$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}$$

- (ii) Use Euclidean algorithm to find two positive integers u and v such that $1269u + 297v = 135$. 6

- (c) (i) If x, y, z be any three real numbers then show that 6

$$\frac{x^2 + y^2}{x + y} + \frac{y^2 + z^2}{y + z} + \frac{z^2 + x^2}{z + x} \geq x + y + z.$$

- (ii) A linear mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by 6

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, 2x_1 + 3x_2 + x_3, 3x_1 + x_2 + 2x_3),$$

$$(x_1, x_2, x_3) \in \mathbb{R}^3. \text{ Find the matrix of } T \text{ relative to the ordered basis}$$

$$\{(-1, 1, 1), (1, -1, 1), (1, -1, 1)\} \text{ of } \mathbb{R}^3.$$

- (d) (i) State and prove fundamental theorem of equivalence relation. 6

- (ii) Solve by Cardan's method the equation: $x^3 - 12x + 65 = 0$ 6

GE3

DIFFERENTIAL EQUATION AND VECTOR CALCULUS

GROUP-A

1. Answer any *four* questions from the following: 3×4 = 12

- (a) Show that $x=1$ is a regular singular point of the ordinary differential equation

$$(x^2 - 1) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0.$$

- (b) Define Wronskian of n functions. Show that the solutions e^{2x} and xe^{2x} of $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0$ are linearly independent.

- (c) If $\vec{F}(t) = 5t^2 \hat{i} + t \hat{j} - t^3 \hat{k}$, then find $\int_1^2 \vec{F}(t) \times \frac{d^2 \vec{F}(t)}{dt^2} dt$.

- (d) Show that a function f given by $f(x, y) = x^2 |y|$ satisfies Lipschitz condition on $R: |x| \leq 1, |y| \leq 1$.

- (e) Determine whether $x = 0$ is an ordinary point or a regular singular point of the differential equation $2x^2 \left(\frac{d^2 y}{dx^2} \right) + 7x(x+1) \frac{dy}{dx} - 3y = 0$.
- (f) If $\vec{\alpha} \cdot \vec{\beta} = 0$ and $\vec{\alpha} \times \vec{\beta} = \vec{0}$ and $|\vec{\alpha}| \neq 0$, then prove that $\vec{\beta} = \vec{0}$.

GROUP-B

2. Answer any **four** questions from the following:

6×4 = 24

- (a) Solve: $(D^3 - D^2 + 4D - 4)y = \sin 3x$
- (b) Solve: $\frac{dx}{dt} - \frac{dy}{dt} + 3x = \sin t$, $\frac{dx}{dt} + y = \cos t$; given that $x = 1$, $y = 0$ for $t = 0$.
- (c) Using the method of undetermined coefficients, solve $(D^2 + 4)y = x^2 \sin 2x$.
- (d) Solve by the method of variation of parameters: $x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 3y = 2x^4 e^x$.
- (e) Prove that $\vec{r} \cdot \frac{d\vec{r}}{dt} = |\vec{r}| \frac{d}{dt} (|\vec{r}|)$.
- (f) The velocity of a moving particle is given by $\vec{V}(t) = -3t \hat{i} + \sin^2 t \hat{j} + \cos^2 t \hat{k}$. Find the particle position as a function of t , if the position at time $t = 0$ is $\vec{r} = \hat{j}$.

GROUP-C

3. Answer any **two** questions from the following:

12×2 = 24

- (a) (i) Solve: $(D^3 + 3D^2 - 4)y = xe^{-2x}$ 6
- (ii) If $\vec{\alpha} \times \vec{\beta} + \vec{\beta} \times \vec{\gamma} + \vec{\gamma} \times \vec{\alpha} = \vec{0}$, then show that the vectors $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ are coplanar. 3
- (iii) Show that the volume of the tetrahedron whose vertices are given by the vectors $(-\hat{i} + \hat{j} + \hat{k})$, $(\hat{i} - \hat{j} + \hat{k})$, $(\hat{i} + \hat{j} - \hat{k})$ with reference to the fourth vertex as origin is $\frac{2}{3}$ cubic unit. 3
- (b) (i) If $\vec{V} = xy\hat{i} - z^2\hat{j} + xyz\hat{k}$, then show that $\int_C \vec{V} \cdot d\vec{r} = \frac{1}{3}$, when the integral is taken from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$. 6
- (ii) Prove that $\vec{\nabla} \cdot \left\{ \frac{f(r)}{r} \vec{r} \right\} = \frac{1}{r^2} \frac{d}{dr} (r^2 f)$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $|\vec{r}| = r = (x^2 + y^2 + z^2)^{1/2}$. 6
- (c) (i) Solve: $(D^2 - 5D + 6)y = e^{4x}(x^2 + 9)$ 6
- (ii) Solve: $(D^3 + 3D^2 - 4)y = xe^{-2x}$ 6
- (d) (i) Solve the equation $\frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + 3y = 0$ near the ordinary point $x = 0$. 6
- (ii) Solve $x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = 10(x + \frac{1}{x})$. 6

GE4

GROUP-A

1. Answer any **four** questions from the following: 3×4 = 12
- (a) Define order of a permutation. Find the order of the permutation 3
- $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 1 & 3 & 2 & 8 & 6 & 7 \end{pmatrix}.$$
- (b) State Second Isomorphism Theorem. 3
- (c) Find the units and the idempotent elements in $(\mathbb{Z}_6, \cdot)$ and $(\mathbb{Z}_5, \cdot)$. 3
- (d) How many binary compositions can be defined on a set of two elements? Justify your answer. 3
- (e) Let G be a group and $a \in G$. If $o(a)$ be infinite and p be a positive integer, then prove that $o(a^p)$ is also infinite. 3
- (f) Define group homomorphism. Find a homomorphism from $(\mathbb{Z}, +)$ into $(2\mathbb{Z}, +)$. 3

GROUP-B

2. Answer any **four** questions from the following: 6×4 = 24
- (a) Let G be a group such that $(ab)^3 = a^3b^3 \forall a, b \in G$. Prove that $H = \{x^3 : x \in G\}$ is a normal subgroup of G . 6
- (b) Let $f : G \rightarrow G'$ be a group homomorphism. If H' be a subgroup of G' then prove that $f^{-1}(H') = \{a \in G \mid f(a) \in H'\}$ is a subgroup of G . Further prove that, if H' is normal in G' , then $f^{-1}(H')$ is normal in G . 6
- (c) For $n > 1$, prove that the number of even permutations and odd permutations in S_n are equal. 6
- (d) Let a be an element of a group $(G, \circ)$. Show that for integers m, n ; $a^{m+n} = a^m \circ a^n$. 6
- (e) Let $(G, \circ)$ be a finite cyclic group generated by $a \in G$. Show that $o(G) = n$ iff $o(a) = n$. 6
- (f) Let $A(G)$ be the set of all automorphisms of a group G . Show that $A(G)$ forms a non-abelian group under the mapping composition. 6

GROUP-C

3. Answer any **two** questions from the following: 12×2 = 24
- (a) (i) Prove that any group containing four elements must be a commutative group. 6
- (ii) Let G be a group of prime order. Show that G has no proper subgroup. 3
- (iii) Let H be a subgroup of a group G . Let $a, b \in G$. Prove that $aH = bH$ iff $a^{-1}b \in H$. 3

- (b) (i) Let $G = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} ; b \neq 0\}$. Prove that $(G, \circ)$ is an abelian group, where 'o' is defined by $(a, b) \circ (c, d) = (ad + bc, bd) \forall (a, b), (c, d) \in G$. ($\mathbb{Q}$ being the set of rationals). 5
- (ii) Let a be an element of a group $(G, \circ)$. If $o(a) = n$, then $a, a^2, \dots, a^n$ are distinct elements of G . 4
- (iii) Find all elements of order 8 in the group $(\mathbb{Z}_{24}, +)$. 3
- (c) (i) If H be a subgroup of a commutative group G , then prove that the quotient group G/H is commutative. Is the converse true? Justify your answer. 6
- (ii) Let $(G, \circ)$ be a group and $a, b \in G$. If $a \circ b = b \circ a$ and $o(a)$ and $o(b)$ are prime to each other, show that $o(a \circ b) = o(a) \cdot o(b)$. 6
- (d) (i) Prove that every subgroup of a cyclic group is cyclic. 6
- (ii) If G be an infinite cyclic group, then prove that G has exactly two generators. 6

GE5

NUMERICAL METHODS

GROUP-A

1. Answer any **four** questions from the following: 3×4 = 12
- (a) Round off the following numbers to four significant figures: 3
34.46253, 0.700029, 0.0022218
- (b) Prove that $\nabla = 1 - E^{-1}$. 3
- (c) Find a real root of the equation $x = e^{-x}$ using Newton-Raphson method. 3
- (d) Using Euler's method, solve the following problem: 3
$$\frac{dy}{dx} = \frac{3}{5}x^3y, \quad y(0) = 1$$
- (e) Given $f(x) = \frac{1}{x^2}$. Find the divided difference $[a, b]$. 3
- (f) Explain the term 'round off error' with an example. 3

GROUP-B

2. Answer any **four** questions from the following: 6×4 = 24
- (a) The equation $2x = \log_{10} x + 7$ has a root between 3 and 4. Find this root, correct up to three decimal places by Regula-Falsi method. 6
- (b) Find a real root of the equation $x^3 - 2x - 5 = 0$ using Secant method. 6
- (c) Establish Newton's divided difference interpolation formula. 6

- (d) Find $f(0.23)$ from the following table using Newton's forward interpolation formula:

x	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

- (e) Given the set of tabulated points (0, 2), (1, 3), (2, 12) and (15, 3587) satisfying the function $y = f(x)$, compute $f(4)$ using Newton's divided difference formula.

- (f) Given the differential equation $\frac{dy}{dx} = x^2 + y$ with $y(0) = 1$. Compute $y(0.02)$ using Euler's modified method.

GROUP-C

3. Answer any *two* questions from the following:

12×2= 24

- (a) (i) Solve the system $6x + y + z = 20$;
 $x + 4y - z = 6$;
 $x - y + 5z = 7$;

Using Jacobi's method.

- (ii) Compute the value of the integral $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with $h = 0.1$.

- (b) (i) Find the inverse of the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 2 \end{bmatrix}$ using Gauss-Jordan Method.

- (ii) Given $\frac{dy}{dx} = y - x$, $y(0) = 2$. Find $y(0.2)$ correct upto 4 decimal places using Runge-Kutta 2nd order formula.

- (c) (i) Describe Gauss-Seidel method for solving a system of linear equations. Obtain the sufficient condition for convergence of Gauss-Seidel method.

- (ii) Find $\frac{dy}{dx}$ at $x = 1.2$ from the following table:

x	1	2	3	4	5	6
y	1.9866	2.9542	3.8941	4.7942	5.6464	6.4421

- (d) (i) Explain the method of fixed point iteration with the condition of convergence for numerical solution of an equation of the form $x = \phi(x)$.

- (ii) Find the interpolating polynomial of degree three relevant to the following data:

x	-1	0	1	2
$f(x)$	1	1	-2	-5

—x—