



UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 1st Semester Examinations, 2018

GE-MATHEMATICS

Time Allotted: 2 Hours

Full Marks: 60

*The figures in the margin indicate full marks.**Candidates should answer in their own words and adhere to the word limit as practicable.**All symbols are of usual significance.*

The question paper contains GE-I, GE-II, GE-III, GE-IV & GE-V. The candidates are required to answer any *one* from the five courses.

Candidates should mention it clearly on the answer book.

GE-I

CALCULUS, GEOMETRY AND DIFFERENTIAL EQUATION

GROUP-A

1. Answer any *four* questions from the following: 3×4 = 12
- (a) Find y_n , where $y = \frac{1}{x^2 + a^2}$. 3
- (b) Evaluate: $\lim_{x \rightarrow 0} \frac{e^{3x} - 3x - 1}{1 - \cos x}$. 3
- (c) Discuss the nature of the conic represented by - 3
 $18x^2 - 4xy + 11y^2 - 44x - 20y + 71 = 0$
- (d) Solve: $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ 3
- (e) Find the points of inflection of the curve $y = (1+x)\tan^{-1}x$. 3
- (f) Find $\int \tan^5 x dx$. 3

GROUP-B

2. Answer any *four* questions from the following: 6×4 = 24
- (a) Solve the equation $y^2(y - xp) = x^4 p^2$ and find its singular solution. 6
- (b) Determine $\int t^3 e^{2t} dt$ using a reduction formula. 6
- (c) Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is $\frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$. 6

- (d) If $y^{1/m} + y^{-1/m} = 2x$, prove that $(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$. 6
- (e) Determine the order and degree of $\left\{1 + \left(\frac{dy}{dx}\right)^4\right\}^{1/3} = \frac{d^2y}{dx^2}$. Find the differential equation of the family of confocal conics $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$. 2+4
- (f) Determine the asymptotes of the curve $y^3 - x^2y + 2y^2 + 4y + 1 = 0$. 6

GROUP-C

3. Answer any **two** questions from the following: 12×2 = 24
- (a) (i) Solve: $x(x^2 + 1)\frac{dy}{dx} = y(1 - x^2) + x^3 \log x$. 6+6
- (ii) Solve: $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$.
- (b) (i) Evaluate: $\lim_{x \rightarrow \infty} (\cosh^{-1} x - \ln x)$. 4+4+4
- (ii) Find the envelope of the family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the parameters a & b are connected by $a + b = c$, c being a constant.
- (iii) If $y = \cosh(\sin^{-1} x)$, show that $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + 1)y_n = 0$.
- (c) (i) Find the area enclosed between the curve $y = x^3$ and the line $y = x$. 3+3+6
- (ii) Find the equation of the sphere passing through the points $(1, 1, 1)$, $(-2, 1, 2)$, $(3, -3, 1)$ and $(-1, 2, -1)$.
- (iii) Find the equation of the right circular cylinder of radius 3 whose axis passes through the point $(1, 3, 4)$ and has $(1, -2, 3)$ as its direction ratios.
- (d) (i) Evaluate: $\int_0^{\pi/2} \sin^5 x \, dx$. 6+(4+1+1)
- (ii) Reduce the equation $4x^2 - 4xy + y^2 - 8x - 6y + 5 = 0$ to its canonical form and show that it represents a parabola. Find the latus rectum and axis of the parabola.

GE-II

ALGEBRA

GROUP-A

1. Answer any **four** questions from the following: 3×4 = 12
- (a) Prove that $5^{2n} - 1$ is divisible by 24 for all $n \in \mathbb{N}$. 3
- (b) Examine if the relation ρ on $\mathbb{Z}$ defined by $a\rho b$ iff $3a + 4b$ is divisible by 7 is an equivalence relation. 3

- (c) If A and B are two invertible matrices of order n , prove that $(AB)^{-1} = B^{-1}A^{-1}$. 3
- (d) Find the rank of the matrix $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 5 \\ 3 & 7 & 8 \end{pmatrix}$. 3
- (e) Show that the mapping $f: Q \rightarrow Q$ defined by $f(x) = 5x + 2$ ($x \in Q$) is a bijective mapping. 3
- (f) If $T: V_3 \rightarrow V_1$ and $T(x, y, z) = x^2 + y^2 + z^2$, then show that T is not a linear transformation. 3

GROUP-B

2. Answer any *four* questions from the following: $6 \times 4 = 24$
- (a) If $\tan^{-1}(x + iy) = \alpha + i\beta$, where x, y, α, β are real and $(x, y) \neq (0, \pm 1)$, then prove that $x^2 + y^2 + 2x \cot \alpha = 1$. 6
- (b) Find the conditions that the roots of the equation $x^3 - px^2 + qx - r = 0$ will be in arithmetic progression. 6
- (c) If a, b, c, d be positive real numbers, not all equal, prove that $(a + b + c + d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) > 16$. 6
- (d) Show that the set $T = \{(x, y, z) \in \mathbb{R}^3 : x = z = 0\}$ is a subspace of $\mathbb{R}^3$. Find $\dim T$. 3+3
- (e) Show that the mapping $f: \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{1\}$ defined by $f(x) = (x+1)/(x-1)$ is a bijection. Determine f^{-1} . 3+3
- (f) If $x + \frac{1}{x} = 2 \cos \frac{\pi}{7}$, then prove that $x^7 + \frac{1}{x^7} + 2 = 0$. 6

GROUP-C

3. Answer any *two* questions from the following: $12 \times 2 = 24$
- (a) (i) If $\sin(a + ib) = x + iy$, then prove that $x^2/\cosh^2 b + y^2/\sinh^2 b = 1$ and $x^2/\sin^2 a - y^2/\cos^2 a = 1$. $(3+3)+6$
- (ii) Prove that $\tan \left\{ i \log \frac{a - ib}{a + ib} \right\} = \frac{2ab}{a^2 - b^2}$.
- (b) (i) Examine the nature of the intersection of the triad of planes: $2x - y + z = 5, x + 2y + 4z = 7, 5x + 3y - z = 0$. 6+6
- (ii) Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

- (c) (i) Use division algorithm to prove that the square of an odd integer is of the form $8k + 1$, where k is an integer. 4+6+2
- (ii) Solve, by Cardan's method, the equation $x^3 - 18x - 35 = 0$.
- (iii) Prove that 0 is an eigenvalue of a singular matrix.
- (d) (i) If a, b, c are three positive rational numbers then prove that 6+(3+3)
- $$\left(\frac{a^2 + b^2 + c^2}{a + b + c} \right)^{a+b+c} \geq a^a b^b c^c \geq \left(\frac{a + b + c}{3} \right)^{a+b+c}.$$
- (ii) Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 5 & -7 \end{pmatrix}$ and hence solve the following equations: $x + y + z = 8$, $x - y + 2z = 6$, $3x + 5y - 7z = 14$.

GE-III

DIFFERENTIAL EQUATIONS AND VECTOR CALCULUS

GROUP-A

1. Answer any **four** questions from the following: 3×4 = 12
- (a) Define Wronskian for n functions. Show that $\{1, x, x^2\}$ is linearly independent.
- (b) Reduce the equation $y'' - \frac{2}{x}y' - n^2y = 0$ into normal form.
- (c) Solve: $(x^2D^2 - 3xD + 4)y = x^2$.
- (d) Define regular singular point. Find the regular singular points of $(1-t^2)y'' + ty' + 2y = 0$.
- (e) If $\vec{\alpha} = \hat{i} + \hat{j} - 6\hat{k}$; $\vec{\beta} = \hat{i} - 3\hat{j} + 4\hat{k}$; $\vec{\gamma} = 2\hat{i} - 5\hat{j} + 3\hat{k}$, then find $(\vec{\alpha} \times \vec{\beta}) \times \vec{\gamma}$.
- (f) If $\vec{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$, then find $\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}$.

GROUP-B

2. Answer any **four** questions from the following: 6×4 = 24
- (a) Solve: $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 8y = 65 \cos(\log x)$.
- (b) Solve: $(D^3 - 2D^2 - 5D + 6)y = (e^{2x} + 3)^2 + e^{3x} \cosh x$.

- (c) Solve by the method of variation of parameters: $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^2 e^{2x}$.
- (d) Show that linearly independent solutions of $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$ are $e^x \sin x$ and $e^x \cos x$. Find the general solution. Also find the solution $y(x)$ with the conditions $y(0) = 2, \frac{dy(0)}{dx} = -3$.
- (e) Find the volume of the parallelopiped whose sides are given by the vectors $-\hat{i} + \hat{j} + 2\hat{k}, \hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + \hat{j} - \hat{k}$.
- (f) If $\vec{F} = (5x^2 + 6y)\hat{i} - (3x + 2y^2)\hat{j} + 2xz^2\hat{k}$, then evaluate $\int_C \vec{F} \cdot d\vec{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the path C given by $x = t, y = t^2, z = t^3$.

GROUP-C

3. Answer any *two* questions from the following: 12×2 = 24

- (a) (i) Solve: $\frac{dx}{dt} + 4x + 3y = t$; $\frac{dy}{dt} + 2x + 5y = e^t$. 6+6

(ii) Solve the following differential equation by the method of undetermined coefficients: $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = xe^x$.

- (b) (i) Find the power series solution of the equation 8+4
 $\frac{d^2 y}{dx^2} - 2x^2 \frac{dy}{dx} + 4xy = x^2 + 2x + 2$.

(ii) Show that $x = 0$ is an ordinary point of $(x^2 - 1) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0$, but $x = 1$ is a regular singular point.

- (c) (i) Solve: $y''' - 2y'' - 19y' + 20y = xe^x + 2e^{-4x} \sin x$. 6+6

(ii) Define triple product of vectors.

$$\text{Show that } [\vec{\alpha} + \vec{\beta} \quad \vec{\beta} + \vec{\gamma} \quad \vec{\gamma} + \vec{\alpha}] = 2[\vec{\alpha} \quad \vec{\beta} \quad \vec{\gamma}].$$

- (d) (i) Define divergence and curl of a vector. (2+5)+5

$$\text{Prove that } \vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

- (ii) Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2 \hat{i} + xy \hat{j}$ and C is the arc bounded by $y = x^2$ and $x = y^2$.

GE-IV
GROUP THEORY

GROUP-A

1. Answer any **four** questions from the following: 3×4 = 12
- (a) Prove that every group of even order has an element of order 2. 3
 - (b) Find the generators and order of each element of $G = \{a, a^2, a^3, \dots, a^8 = e\}$. 3
 - (c) Prove that the order of an r -cycle is r . 3
 - (d) Prove that if $f: G \rightarrow G'$ is a group isomorphism then $f^{-1}: G' \rightarrow G$ is also a group isomorphism. 3
 - (e) If H is a subgroup of a group G such that $x^2 \in H$ for all $x \in G$, then prove that H is a normal subgroup of G . 3
 - (f) Prove that the centralizer $C(a)$ of an element a of a group G is a subgroup of G . 3

GROUP-B

2. Answer any **four** questions from the following: 6×4 = 24
- (a) If $a, x \in (G, \circ)$, prove that $O(a) = O(x^{-1} \circ a \circ x)$. Hence prove that $O(a \circ b) = O(b \circ a)$, $a, b \in (G, \circ)$. 6
 - (b) Prove that a non-empty subset H of a group $(G, \circ)$ is a subgroup of G if and only if (i) $a, b \in H \Rightarrow a \circ b \in H$ and (ii) $a \in H \Rightarrow a^{-1} \in H$. 6
 - (c) Find the eight symmetries of a square and show that they form a non-abelian group. 6
 - (d) Prove that the set $\mathbb{Z}_n$, the classes of residue of integers modulo n forms an abelian group with respect to addition modulo n . 6
 - (e) (i) If the mapping $f: G \rightarrow G'$ is a group isomorphism, then prove that $O(a) = O(f(a))$ for all $a \in G$. 4+2
(ii) Prove that the inverse of an element in a subgroup of a group G is the same as that element regarded in G .
 - (f) Prove that the set Q of all rational numbers other than 1 with the operation $\circ$ defined by $a \circ b = a + b - ab$ is an abelian group. 6

GROUP-C

3. Answer any **two** questions from the following: 12×2 = 24
- (a) (i) Prove that a cyclic group of prime order has no proper subgroup. 5+2+5
(ii) State third isomorphism theorem for group.
(iii) Let $(G, \circ)$ and $(G', *)$ be two groups and let $Q: G \rightarrow G'$ be an isomorphism. Prove that if G is commutative then G' is commutative and conversely.

- (b) (i) Let H be a subgroup of a group G . Prove that any two left cosets of H in G are either identical or they have no common element. 5+4+3
- (ii) State and prove Lagrange's Theorem.
- (iii) If H is a subgroup of a group G and N is a normal subgroup of G , then show that $H \cap N$ is a normal subgroup of H .
- (c) (i) Let H and K be two finite cyclic group of order m and n respectively. Prove that internal direct product $H \times K$ is cyclic if and only if $\gcd(m, n) = 1$. 7+5
- (ii) Find all the homomorphism of the group $(\mathbb{Z}, +)$ to the group $(\mathbb{Z}, +)$.
- (d) (i) Prove that the intersection of two subgroups of a group $(G, \circ)$ is a subgroup of G . Is it true for union? Justify your answer. 6+3+3
- (ii) Prove that H is a normal subgroup of a group G if and only if $xHx^{-1} \subset H$ for all $x \in G$.
- (iii) Define even permutation. Determine whether the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}$ is odd or even.

GE-V

NUMERICAL METHODS

GROUP-A

1. Answer any *four* questions from the following: 3×4 = 12
- (a) Round off the following numbers correct up to 4-decimal places:
- (i) 56.243827 (ii) 0.560012 (iii) 0.005789
- (b) Find a positive root of $x^2 + 2x - 2 = 0$ by Newton-Raphson method correct up to two significant figures.
- (c) Given that $\frac{dy}{dx} = x + y$ with the initial condition $y(0) = 1$. Compute $y(0.1)$ by Euler's method taking $h = 0.05$.
- (d) Prove that $(1 + \nabla)(1 - \nabla)f(x) = f(x)$.
- (e) What is meant by complete pivoting and partial pivoting in case of solving a set of linear equations by Gauss elimination method?
- (f) Obtain Trapezoidal rule from Newton-Cotes formula.

GROUP-B

2. Answer any *four* questions from the following: 6×4 = 24

- (a) Find the value of $f(2.5)$ from the following data by using suitable interpolation formula: 6

x	2	3	4	5
$f(x)$	14.5	16.3	17.5	18.0

- (b) Evaluate $y(0.02)$ given $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ by modified Euler's method. 6

- (c) Solve the system of linear equations by Gauss-Jordan method: 6

$$2x + y + z = 0 ; 3x + 2y + 3z = 0 ; x + 4y + 9z = 16.$$

- (d) Find an estimate for the error of Newton-Raphson method and show that it has quadratic rate of convergence. 6

- (e) What do you understand by open and closed type numerical integration formula? Obtain Simpson's 1/3 rule for numerical integration in composite form. Write down its error formula. 1+4+1

- (f) In usual notation, show that $\Delta^m \left(\frac{1}{x} \right) = \frac{(-1)^m m! h^m}{x(x+h)(x+2h)\dots(x+mh)}$. 6

GROUP-C

3. Answer any *two* questions from the following: 12×2 = 24

- (a) What do you mean by 'round-off error' in numerical data? Show how this errors are propagated in a difference table? Solve by Gauss-Seidal iterative method: 1+4+7

$$3x + 9y - 2z = 11 ; 4x + 2y - 13z = 24 ; 4x - 2y + z = -8.$$

- (b) (i) Compute $\int_1^4 \frac{\log_e(1+0.5x+x^2)}{0.5+x} dx$ by Trapezoidal rule taking 13 ordinates, correct up to 6 decimal places. 6+6

- (ii) Solve $x^2 + 2x - 1 = 0$ by Bisection method, correct to 5-significant figures.

- (c) (i) A function $f(x)$ defined on $[0, 1]$ is such that $f(0) = 0$, $f(1/2) = -1$, $f(1) = 0$. Find the interpolation polynomial $P(x)$ approximating $f(x)$. 5+7

- (ii) Establish Lagrange's interpolation formula for $(n+1)$ distinct arguments in an interval $[a, b]$.

- (d) (i) Explain Runge-Kutta method of order two for solving the differential equation $\frac{dy}{dx} = f(x, y)$. 6+6

- (ii) Prove that the remainder in approximating $f(x)$ by the interpolation polynomial using interpolating points $x_0, x_1, x_2, \dots, x_n$ is of the form

$$(x-x_0)(x-x_1)(x-x_2)\dots(x-x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!}, \text{ where } \xi \text{ is a number lying}$$

between the smallest and greatest of $x_0, x_1, x_2, \dots, x_n$.

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